ELEN E3401: Electromagnetics

Spring 2025

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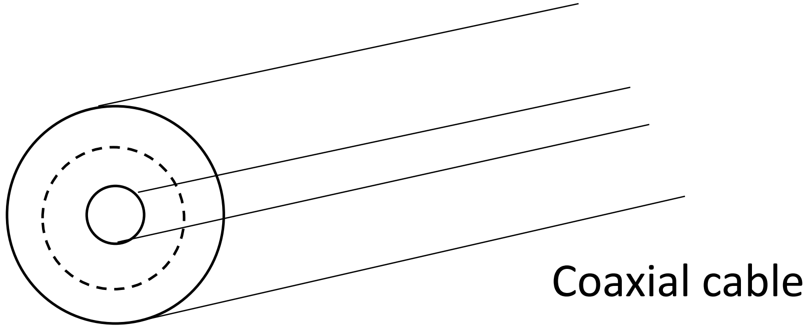
Lecture #4

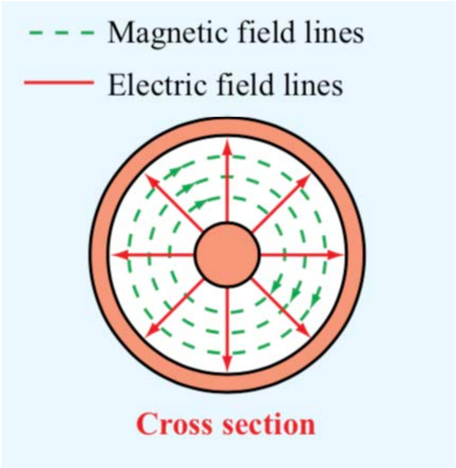
COLUMBIA | ENGINEERING
The Fu Foundation School of Engineering and Applied Science

Propagation Modes

TEM (Transverse Electromagnetic):

Electric and magnetic fields are orthogonal to one another, and both are orthogonal to direction of propagation





- Higher order transmission line: waves have significant field component along direction of propagation

- Examples: optical fibers, rectangular waveguides

TEM Lines

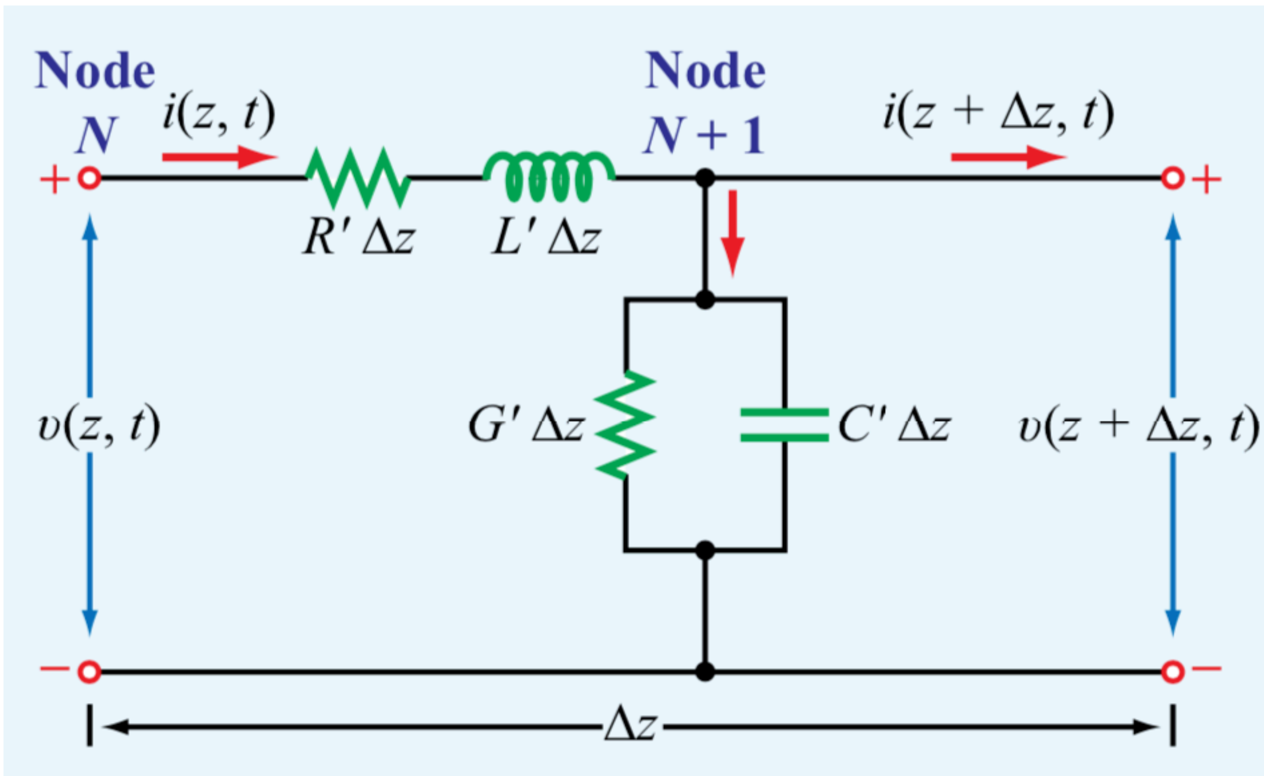
- TEM – transverse electromagnetic modes

- For <u>all</u> TEM Lines:
  - $L'C' = \mu\epsilon$
  - $\frac{G'}{C'} = \frac{\sigma}{\epsilon}$

- For air line (insulating medium = air)
  - $\epsilon = \epsilon_0$
  - $\mu = \mu_0$
  - $\sigma = 0 \rightarrow \sigma' = 0$

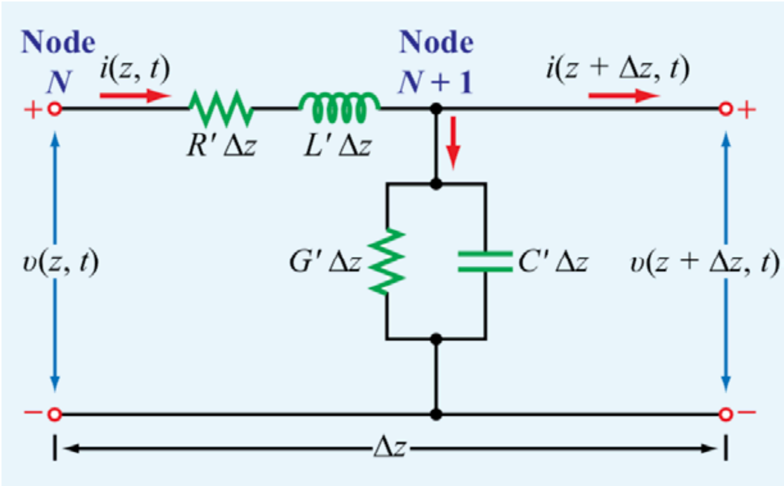
Transmission Line Equations

We develop model for voltage and current
across differential length → Δz of transmission line:



Lumped element model for differential length Δz

Transmission Line Equations



Apply KVL:

$$v(z,t) - i(\mathrm{z,t})R'\Delta\mathrm{z} - \mathrm{L}'\Delta\mathrm{z}\frac{\partial i(z,t)}{\partial t} - v(z+\Delta\mathrm{z},t) = 0$$

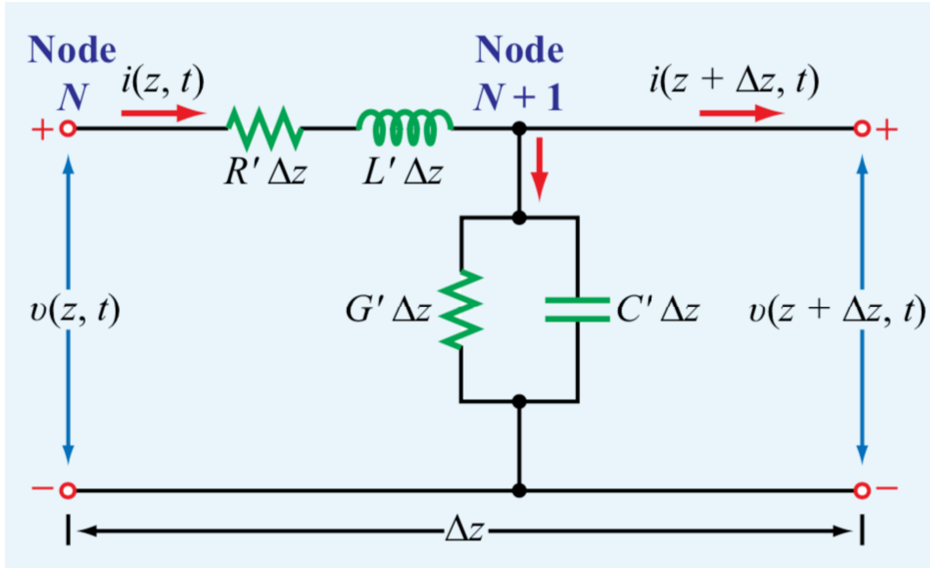
Divide by Δz:

$$-\left[\frac{v(z+\Delta\mathrm{z},t) - v(z,t)}{\Delta\mathrm{z}}\right] = \mathrm{R}'i(\mathrm{z,t}) + \mathrm{L}'\frac{\partial i(z,t)}{\partial t}$$

becomes differential equation Δz → 0

$$\boxed{-\frac{\partial v(z,t)}{\partial z} = \mathrm{R}'i(\mathrm{z,t}) + L'\frac{\partial i(z,t)}{\partial t}} \quad 1$$

Transmission Line Equations



Apply KCL at (N+1):

$$i(z,t) - G'\Delta z\, v(z+\Delta z,t) - C'\Delta z \frac{\partial v(z+\Delta z,t)}{\partial t} - i(z+\Delta z,t) = 0$$

Divide by Δz, then $\Delta z \rightarrow 0$:

$$\boxed{-\frac{\partial i(z,t)}{\partial z} = G' v(z,t) + C' \frac{\partial v(z,t)}{\partial t}} \quad 2$$

Transmission Line Equations

Equations 1 and 2

→ time domain transmission line equations, 'telegrapher equations'

$$\boxed{-\frac{\partial v(z,t)}{\partial z} = \mathrm{R}' i(\mathrm{z},\mathrm{t}) + L' \frac{\partial i(z,t)}{\partial t}} \qquad (1)$$

$$\boxed{-\frac{\partial i(z,t)}{\partial z} = \mathrm{G}' v(\mathrm{z},\mathrm{t}) + C' \frac{\partial v(z,t)}{\partial t}} \qquad (2)$$

We will look at the <u>sinusoidal steady-state</u>

Transmission Line Equations (Phasor)

Recall:

$$v(z,t) = Re\left[\tilde{V}(z)e^{j\omega t}\right]$$

$$i(z,t) = Re\left[\tilde{I}(z)e^{j\omega t}\right]$$

$$\frac{d}{dt} \leftarrow\rightarrow j\omega$$

$$\boxed{-\frac{\partial v(z,t)}{\partial z} = \mathrm{R}' i(\mathrm{z},\mathrm{t}) + L'\frac{\partial i(z,t)}{\partial t}} \quad (1)$$

$$\boxed{-\frac{\partial i(z,t)}{\partial z} = \mathrm{G}' v(\mathrm{z},\mathrm{t}) + C'\frac{\partial v(z,t)}{\partial t}} \quad (2)$$

$$\left.\begin{aligned} -\frac{d\tilde{V}(z)}{dz} &= (\mathrm{R}'+j\omega L')\tilde{I}(z) \\ -\frac{d\tilde{I}(z)}{dz} &= (\mathrm{G}'+j\omega C')\tilde{V}(z) \end{aligned}\right\} \text{Phasor form}$$

Wave Propagation in Transmission Line

We can take $\frac{d}{dz}$ of 1st phasor equation:

$$-\frac{d^2\tilde{V}(z)}{dz^2} = (\mathrm{R}'+j\omega L')\frac{d\tilde{I}(z)}{dz}$$

Substitute $-\frac{d\tilde{I}(z)}{dz} = (\mathrm{G}'+j\omega C')\tilde{V}(z)$ (from 2nd phasor equation) into the above equation

$$\frac{d^2\tilde{V}(z)}{dz^2} - (\mathrm{R}'+j\omega L')(\mathrm{G}'+j\omega C')\tilde{V}(z) = 0$$

$$\boxed{\frac{d^2\tilde{V}(z)}{dz^2} - \gamma^2\tilde{V}(z) = 0}$$ ← Wave equation $\tilde{V}(z)$

Where $\gamma = \sqrt{(\mathrm{R}'+j\omega L')(\mathrm{G}'+j\omega C)}$ is the complex **propagation constant**

Wave Propagation in Transmission Line

Similarly, obtain wave equation for $\tilde{I}(z)$

$$\frac{d^2\tilde{I}(z)}{dz^2} - \gamma^2\tilde{I}(z) = 0 \leftarrow \text{Wave equation } \tilde{I}(z)$$

$$\gamma = complex\ propagation\ constant\ of\ TL$$

$\gamma = \alpha + j\beta \longleftarrow$ Imaginary part → phase constant (rad/m)

Real part → attenuation constant (Np/m)

$$\alpha = Re\left[\sqrt{(\mathrm{R}'+j\omega L')(\mathrm{G}'+j\omega C)}\right]$$

$$\beta = Im[\sqrt{(\mathrm{R}'+j\omega L')(\mathrm{G}'+j\omega C)}]$$

* Note: Passive transmission lines → α is ≥ 0
(active → gain of laser)

Traveling wave solutions

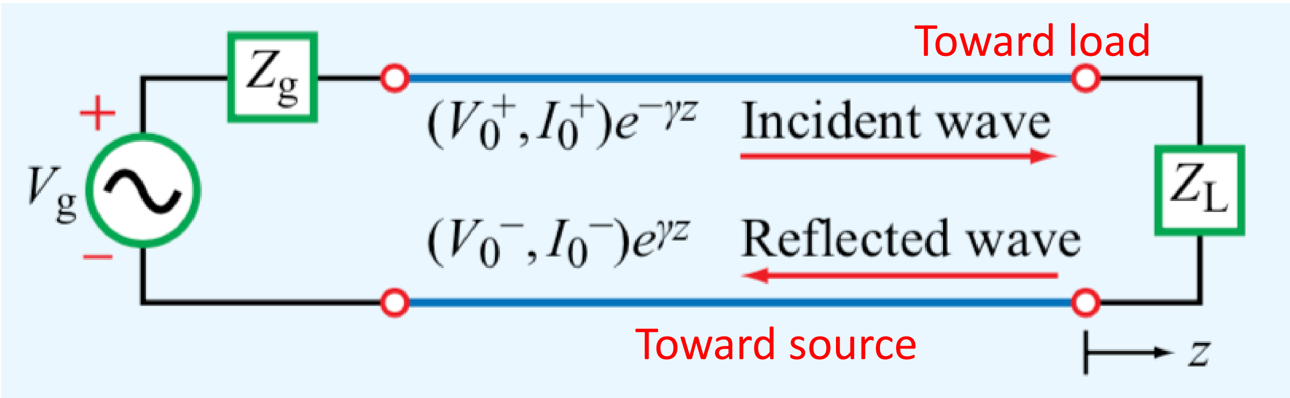
Traveling wave solutions:

$$\begin{cases} \widetilde{V}(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \\ \tilde{I}(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z} \end{cases}$$

Verify solutions by inserting into wave equations
(what are the + and – propagation directions):

$$\begin{cases} e^{-\gamma z} \rightarrow \text{wave propagating +z direction} \\ e^{+\gamma z} \rightarrow \text{wave propagating -z direction} \end{cases}$$

Traveling wave solutions



4 unknowns:
(V_0^+, I_0^+) and (V_0^-, I_0^-)

Wave amplitudes of incident/reflected

Relate the 4 wave amplitudes: I_0^+ and I_0^- to V_0^+ and V_0^-

$$\tilde{V}(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$ Total voltage

$$\tilde{I}(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}$$ Total current

$$-\frac{d\tilde{V}(z)}{dz} = (\mathrm{R}' + j\omega L')\tilde{I}(z)$$ ← Transmission line voltage phasor equation

$$-\frac{d\tilde{V}(z)}{dz} = \gamma V_0^+ e^{-\gamma z} - \gamma V_0^- e^{\gamma z} = (\mathrm{R}' + j\omega L')\tilde{I}(z)$$

$$\Rightarrow \quad \tilde{I}(z) = \frac{\gamma}{R' + j\omega L'}[V_0^+ e^{-\gamma z} - V_0^- e^{\gamma z}].$$

Characteristic Impedance

$$\tilde{I}(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z} \qquad \tilde{I}(z) = \frac{\gamma}{R' + j\omega L'}\left[V_0^+ e^{-\gamma z} - V_0^- e^{\gamma z}\right]$$

$$I_0^+ = \frac{V_0^+ \gamma}{(\mathrm{R}' + j\omega L')} \quad and \quad I_0^- = \frac{-V_0^- \gamma}{(\mathrm{R}' + j\omega L')}$$

- Z_0 is the characteristic impedance
- Z_0 is the ratio of voltage amplitude to current amplitude for each individual traveling wave
- *Not* ratio of total $\tilde{V}(z)$ and $\tilde{I}(z)$

Define the ratio: $\dfrac{V_0^+}{I_0^+} = Z_0 = \dfrac{-V_0^-}{I_0^-}$

Characteristic impedance

$$Z_0 = \frac{R' + j\omega L'}{\gamma} = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}} \quad \Omega$$